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ON SOME PROBABILISTIC ASPECTS OF (GENERALIZED) DICYCLIC GROUPS

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Abstract. In this paper we study probabilistic aspects, such as the subgroup commutativity degree and the cyclic subgroup commutativity degree, of (generalized) dicyclic groups. We find explicit formulas for these quantities and we provide another example of a class of groups whose (cyclic) subgroup commutativity degree vanishes asymptotically.

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Key words: Subgroup commutativity degree, subgroup lattice, cyclic subgroup commutativity degree, poset of cyclic subgroups.

1. Introduction. The connection between finite group theory and probabilities is a topic which is frequently studied in the last years. Given a finite group G, many results regarding the membership of G to a class of groups were obtained using a relevant concept known as the *commutativity degree* of G, which measures the probability that two elements of the group commute. We refer the reader to [2]–[4] and [7]–[11] for more details.

Inspired by this idea, in [17, 20], other two probabilistic aspects called the *subgroup commutativity degree* and the *cyclic subgroup commutativity degree* of a finite group G were introduced. Denoting by $L(G)$ and $L_1(G)$ the subgroup lattice and the poset of cyclic subgroups of G, respectively, the above concepts are defined by

$$\begin{aligned} sd(G) &= \frac{1}{|L(G)|^2}|\{(H,K) \in L(G)^2 \mid HK = KH\}| \\ &= \frac{1}{|L(G)|^2}|\{(H,K) \in L(G)^2 \mid HK \in L(G)\}| \end{aligned}$$

and

$$\begin{aligned} csd(G) &= \frac{1}{|L_1(G)|^2}|\{(H,K) \in L_1(G)^2 \mid HK = KH\}| \\ &= \frac{1}{|L_1(G)|^2}|\{(H,K) \in L_1(G)^2 \mid HK \in L(G)\}|, \end{aligned}$$

respectively. In other words, the (cyclic) subgroup commutativity degree measures the probability that two (cyclic) subgroups of G commute. Moreover, for a subgroup H of G, by introducing the sets

$$C(H) = \{K \in L(G) \mid HK = KH\}$$

and

$$C_1(H) = \{K \in L_1(G) \mid HK = KH\},$$

we obtain

$$sd(G) = \frac{1}{|L(G)|^2} \sum_{H \in L(G)} |C(H)|$$

and

$$csd(G) = \frac{1}{|L_1(G)|} \sum_{H \in L_1(G)} |C_1(H)|.$$

In [1, 6, 13, 17, 18, 20], explicit formulas for $sd(G)$ and $csd(G)$, where G belongs to some particular classes of groups, were provided. In the same papers, another purpose was to study the asymptotic behaviour of the (cyclic) subgroup commutativity degree. In other words, given a class of finite groups $\{G_n\}_{n\geq 1}$, what can be said about the limits $\lim\limits_{n\to\infty} sd(G_n)$ and $\lim\limits_{n\to\infty} csd(G_n)$? In this paper, we analyse the same problems concerning the dicyclic groups and some generalized dicyclic groups.

For a positive integer $n \geq 1$, the dicyclic group, denoted by Dic_{4n}, is defined as

$$Dic_{4n} = \langle a, \gamma \mid a^{2n} = e, \gamma^2 = a^n, a^\gamma = a^{-1}\rangle.$$

We mention that, by definition, $Dic_4 \cong \mathbb{Z}_4$. Hence, $sd(Dic_4) = csd(Dic_4) = 1$, so we are especially interested in the case $n \geq 2$. We note that, for dicyclic groups, the number of (cyclic) factorizations, which is a concept that is connected with the (cyclic) subgroup commutativity degree, was studied in [12, 21]. The dicyclic group has the following generalization: for an arbitrary abelian group A of order $2n$, a generalized dicyclic group, denoted by $Dic_{4n}(A)$, is

$$Dic_{4n}(A) = \langle A, \gamma \mid \gamma^4 = e, \gamma^2 \in A \setminus \{e\}, g^\gamma = g^{-1},\ \forall\, g \in A\rangle.$$

It is easy to see that $Dic_{4n}(\mathbb{Z}_{2n}) = Dic_{4n}$. Concerning the study of the subgroup commutativity degree and of the cyclic subgroup commutativity degree of $Dic_{4n}(A)$, we mention that we will limit to the case $A \cong \mathbb{Z}_2 \times \mathbb{Z}_n$, where n is a positive even integer. To explain this choice, let a and b the generators of the groups $\mathbb{Z}_n$ and $\mathbb{Z}_2$, respectively. Since $\gamma^4 = e$ and $\gamma^2 \in A \setminus \{e\}$, we deduce that

γ^2 is an element of order 2 that is contained in A. Hence, we have only 3 possible choices for γ^2, these being $a^{\frac{n}{2}}, b$ and $a^{\frac{n}{2}}b$. Note that, for each of these 3 options, we get a different generalized dicyclic group.

This paper is organised as follows. Section 2 provides explicit formulas for the subgroup commutativity degree of the dicyclic group Dic_{4n} and of the generalized dicyclic groups $Dic_{4n}(\mathbb{Z}_2 \times \mathbb{Z}_n)$. For the latter, we cover a particular case, namely $n = 2^m$, where $m \geq 2$ is a positive integer, which raises the problem of counting the number of subgroups of $\mathbb{Z}_2 \times D_{2^m}$ and of $\mathbb{Z}_2 \times Q_{2^{m+1}}$. Results concerning the cyclic subgroup commutativity degree of the above mentioned groups are outlined in Section 3. Some open problems and further research directions are indicated in the last section.

Most of our notation is standard and will usually not be repeated here. Elementary concepts and results on groups can be found in [5, 15]. For subgroup lattice concepts we refer the reader to [14, 16, 19].

2. Subgroup commutativity degree of (generalized) dicyclic groups. Let $n \geq 1$ be a positive integer. We denote by $\tau(n)$ and $\sigma(n)$ the number of divisors of n and their sum, respectively. Also, we consider the arithmetic function g given by

$$g(n) = n \sum_{\substack{r|n \\ s|n}} \frac{1}{(r,s)}, \ \forall\, n \geq 1,$$

which is multiplicative. Moreover, for the prime factorization of n, i.e. $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, we have

$$g(n) = \prod_{i=1}^{k} \frac{(2\alpha_i + 1)p_i^{\alpha_i+2} - (2\alpha_i + 3)p_i^{\alpha_i+1} + p_i + 1}{(p_i - 1)^2}.$$

Our first purpose is to compute explicitly the subgroup commutativity degree of the dicyclic group Dic_{4n}. Recall that the structure of this group is

$$Dic_{4n} = \langle a, \gamma \mid a^{2n} = e, \gamma^2 = a^n, a^\gamma = a^{-1} \rangle.$$

To determine the subgroup commutativity degree of Dic_{4n}, it is useful to know which are its subgroups. For each divisor r of $2n$, the dicyclic group has one subgroup isomorphic to $\mathbb{Z}_r$, namely $H_0^r = \langle a^{\frac{2n}{r}} \rangle$. Also, for each divisor s of n, Dic_{4n} possesses $\frac{n}{s}$ subgroups isomorphic to Dic_{4s}, namely $H_i^s = \langle a^{\frac{n}{s}}, a^{i-1}\gamma \rangle$, where $i \in \{1, 2, \dots, \frac{n}{s}\}$.

THEOREM 2.1. *Let $n = 2^m m' \geq 2$, where $m \in \mathbb{N}$ and m' is a positive odd number. The subgroup commutativity degree of the dicyclic group Dic_{4n} is*

$$sd(Dic_{4n}) = \frac{(m+2)^2\tau(m')^2 + 2(m+2)\tau(m')\sigma(n) + [(m-1)2^{m+3} + 9]g(m')}{[(m+2)\tau(m') + \sigma(n)]^2}.$$

In particular, we have

$$sd(Dic_{12}) = \frac{29}{32}.$$

Proof. The subgroup commutativity degree of Dic_{4n} is given by

$$sd(Dic_{4n}) = \frac{1}{|L(Dic_{4n})|^2}\left(\sum_{r|2n}|C(H_0^r)| + \sum_{s|n}\sum_{i=1}^{\frac{n}{s}}|C(H_i^s)|\right). \tag{1}$$

By inspecting the above mentioned subgroup structure of the dicyclic group, we infer that

$$|L(Dic_{4n})| = \tau(2n) + \sigma(n) = (m+2)\tau(m') + \sigma(n).$$

Let r be a divisor of $2n$. The nature of the subgroup H_0^r and the equality $a^\gamma = a^{-1}$, leads us to

$$|C(H_0^r)| = |L(Dic_{4n})| = (m+2)\tau(m') + \sigma(n).$$

Hence,

$$\sum_{r|2n}|C(H_0^r)| = (m+2)^2\tau(m')^2 + (m+2)\tau(m')\sigma(n).$$

Now, let s be a divisor of n and consider the dicylic subgroup H_i^s of Dic_{4n}, where $i \in \{1, 2, \ldots, \frac{n}{s}\}$. Then

$$C(H_i^s) = \left(\bigcup_{r|2n}\{H_0^r\}\right) \cup \left\{H_j^t \mid H_i^s H_j^t = H_j^t H_i^s, where\ t|n, j \in \left\{1, 2, \ldots, \frac{n}{t}\right\}\right\}.$$

It is known that $Z(Dic_{4n}) = \langle a^n\rangle$. Also, it is clear that $Z(Dic_{4n}) \subset H_i^s$, for each divisor s of n and for all $i \in \{1, 2, \ldots, \frac{n}{s}\}$. Finally, we remark that $\frac{Dic_{4n}}{Z(Dic_{4n})} \cong D_{2n}$, where

$$D_{2n} = \langle x, y \mid x^n = y^2 = e, x^y = x^{-1}\rangle$$

is the dihedral group of order $2n$. Thus,

$$H_j^t \in C(H_i^s) \iff H_i^s H_j^t = H_j^t H_i^s \iff \frac{H_i^s}{Z(Dic_{4n})}\frac{H_j^t}{Z(Dic_{4n})} = \frac{H_j^t}{Z(Dic_{4n})}\frac{H_i^s}{Z(Dic_{4n})}.$$

Notice that, by the isomorphism $\frac{Dic_{4n}}{Z(Dic_{4n})} \cong D_{2n}$, the subgroups $\frac{H_i^s}{Z(Dic_{4n})}$ and $\frac{H_j^t}{Z(Dic_{4n})}$ of $\frac{Dic_{4n}}{Z(Dic_{4n})}$ are isomorphic with the subgroups $K_i^s = \langle x^{\frac{n}{s}}, x^{i-1}y\rangle$ and $K_j^t = \langle x^{\frac{n}{t}}, x^{j-1}y\rangle$ of D_{2n}. Hence, the above equivalence actually says that the subgroups H_i^s and H_j^t of Dic_{4n} commute if and only if the subgroups K_i^s and K_j^t of D_{2n} commute. This problem was studied in [17] (see Subsection 3.1), and, according to this reference, we obtain

$$\sum_{s|n}\sum_{i=1}^{\frac{n}{s}}|C(H_i^s)| = (m+2)\tau(m')\sigma(n) + [(m-1)2^{m+3} + 9]g(m').$$

Since we determined all the unknown quantities that appear in (1), our proof is complete. □

We continue our study by finding the subgroup commutativity degree of the generalized dicyclic groups $Dic_{4n}(A)$, where $A \cong \mathbb{Z}_2 \times \mathbb{Z}_n$ and n is a positive even integer. We recall that the structure of this group is

$$Dic_{4n}(A) = \langle A, \gamma \mid \gamma^4 = e, \gamma^2 \in A \setminus \{e\}, g^\gamma = g^{-1}, \forall\ g \in A\rangle.$$

Let a and b be the generators of the cyclic groups $\mathbb{Z}_n$ and $\mathbb{Z}_2$, respectively. As we explained in the first section, we have $\gamma^2 \in \{a^{\frac{n}{2}}, b, a^{\frac{n}{2}}b\}$. In the proof of the main result, which provides the explicit formula of $sd(Dic_{4n}(A))$, we need some preliminary results to analyse the case described by $n = 2^m$, where $m \geq 2$ is a positive integer and $\gamma^2 = a^{\frac{n}{2}}$. We denote by n' the quantity $\frac{n}{2} = 2^{m-1}$. The necessary tools are given by the following lemma and corollary.

Lemma 2.2. *The number of subgroups of the direct product $\mathbb{Z}_2 \times D_{2n'}$ is given by*

$$|L(\mathbb{Z}_2 \times D_{2n'})| = 5\sigma(n') + 3\tau(n') - 2n' - 1.$$

Proof. We apply Goursat's Lemma to prove the desired equality. This result states that there is a bijection between the set of subgroups of $\mathbb{Z}_2 \times D_{2n'}$ and the set of 5-tuples $(H_1, H_1', H_2, H_2', \phi)$, where $H_1 \lhd H_1' \leqslant \mathbb{Z}_2$, $H_2 \lhd H_2' \leqslant D_{2n'}$ and $\phi : \frac{H_1'}{H_1} \longrightarrow \frac{H_2'}{H_2}$ is an isomorphism. Moreover, a subgroup H of $\mathbb{Z}_2 \times D_{2n'}$ is $H = \{(g_1, g_2) \in \mathbb{Z}_2 \times D_{2n'} \mid \phi(g_1H_1) = g_2H_2\}$.

Since $|L(D_{2n'})| = \tau(n') + \sigma(n')$, there are $2(\tau(n') + \sigma(n'))$ subgroups of $\mathbb{Z}_2 \times D_{2n'}$ corresponding to the 5-tuples $(H_1, H_1, H_2, H_2, \phi)$. For each divisor $d \neq n'$ of n', we get a 5-tuple $(\{1\}, \mathbb{Z}_2, \langle x^{\frac{2n'}{d}}\rangle, \langle x^{\frac{n'}{d}}\rangle, \phi)$, where $\{1\}$ denotes the trivial subgroup of $\mathbb{Z}_2$. Hence, we obtain other $\tau(n') - 1$ subgroups of $\mathbb{Z}_2 \times D_{2n'}$. Going further, it is easy to remark that n' subgroups of our direct product are in bijection with the 5-tuples $(\{1\}, \mathbb{Z}_2, \{e\}, \langle x^{i-1}y\rangle, \phi)$, $i \in \{1, 2, \ldots, n'\}$.

To count the number of the remaining subgroups of $\mathbb{Z}_2 \times D_{2n'}$, we take $d \neq 1$ to be a divisor of n'. Remark that for obtaining the isomorphism ϕ, a dihedral subgroup of $D_{2n'}$, namely $\langle x^{\frac{n'}{d}}, x^{i-1}y\rangle$, where $i \in \{1, 2, \ldots, \frac{n'}{d}\}$ is fixed, could be factored by $\langle x^{\frac{n'}{d}}\rangle$ and $\langle x^{\frac{2n'}{d}}, x^{j-1}y\rangle$, where $j \in \{i, i + \frac{n'}{d}\}$. Consequently, we have

$$\sum_{\substack{d|n' \\ d\neq 1}} \sum_{i=1}^{\frac{n'}{d}} 3 = 3(\sigma(n') - n')$$

additional subgroups of our direct product. By inspecting the lattice of subgroups of the dihedral group $D_{2n'}$, we specify that, using its subgroups, we cannot have other possibilities to form a factor group whose order is less than or equal to 2. Hence, we cannot find other 5-tuples. Our proof is complete once we add up the above obtained values. $\square$

Recall that the generalized quaternion group has the following structure

$$Q_{4n'} = \langle x, y \mid x^n = y^4 = 1, x^y = x^{-1}\rangle.$$

It is known that $Q_{4n'}$ has an unique minimal subgroup, which is its center $Z(Q_{4n'}) = \langle x^{n'}\rangle$, and $\frac{Q_{4n'}}{Z(Q_{4n'})} \cong D_{2n'}$. We infer that

$$\frac{\mathbb{Z}_2 \times Q_{4n'}}{Z(Q_{4n'})} \cong \mathbb{Z}_2 \times D_{2n'}.$$

Using the Correspondence Theorem and the fact that there are 3 subgroups of $\mathbb{Z}_2 \times Q_{4n'}$ which do not contain $Z(Q_{4n'})$, namely the trivial subgroup, $\mathbb{Z}_2$ and $\langle x^{n'}b\rangle$, we obtain the following result.

COROLLARY 2.3. *The number of subgroups of the group $\mathbb{Z}_2 \times Q_{4n'}$ is given by*

$$|L(\mathbb{Z}_2 \times Q_{4n'})| = 5\sigma(n') + 3\tau(n') - 2n' + 2.$$

Since $n' = 2^{m-1}$, the number of divisors of n' and their sum are $\tau(n') = m$ and $\sigma(n') = 2^m - 1$, respectively. Then, the conclusion of Corollary 2.3 can be written as

$$|L(\mathbb{Z}_2 \times Q_{2^{m+1}})| = 2^{m+2} + 3(m-1). \tag{2}$$

We are ready to prove the main result concerning the subgroup commutativity degree of the generalized dicyclic groups $Dic_{4n}(\mathbb{Z}_2 \times \mathbb{Z}_n)$.

THEOREM 2.4. *Let $A \cong \mathbb{Z}_2 \times \mathbb{Z}_n$ be an abelian group, where $n = 2^m m' \geq 2$, $m \in \mathbb{N}^*$ and m' is a positive odd integer.*

i) If $m \geq 1, m' \neq 1$ and $\gamma^2 \in \{a^{\frac{n}{2}}, b, a^{\frac{n}{2}}b\}$, or, if $m \geq 2, m' = 1$ and $\gamma^2 \in \{b, a^{\frac{n}{2}}b\}$, the subgroup commutativity degree of the generalized dicyclic group $Dic_{4n}(A)$ is

$$sd(Dic_{4n}(A)) = \frac{|L(A)|^2 + 2|L(A)|\sigma(n) + [(m-1)2^{m+3} + 9]g(m')}{(|L(A)| + \sigma(n))^2}.$$

In particular, we have

$$sd(Dic_{4n}(\mathbb{Z}_2 \times \mathbb{Z}_6)) = \frac{215}{242}.$$

ii) If $m \geq 2, m' = 1$ and $\gamma^2 = a^{\frac{n}{2}}$, the generalized dicyclic group $Dic_{4n}(A)$ is isomorphic to the direct product $\mathbb{Z}_2 \times Q_{2^{m+1}}$ and its subgroup commutativity degree is

$$sd(\mathbb{Z}_2 \times Q_{2^{m+1}}) = \frac{2^{m+2}(24m - 37) + 9m^2 - 18m + 185}{[2^{m+2} + 3(m-1)]^2}.$$

In particular, we have

$$sd(\mathbb{Z}_2 \times Q_{16}) = \frac{333}{361}.$$

iii) If $m = m' = 1$ and $\gamma^2 \in \{a^{\frac{n}{2}}, b, a^{\frac{n}{2}}b\}$, the generalized dicyclic group $Dic_{4n}(A)$ is isomorphic to the abelian group $\mathbb{Z}_2 \times \mathbb{Z}_4$ and its subgroup commutativity degree is

$$sd(\mathbb{Z}_2 \times \mathbb{Z}_4) = 1.$$

Proof. i) Let $n = 2^m m' \geq 2$, such that $m \geq 1, m' \neq 1$, or, $m \geq 2, m' = 1$, and $\gamma^2 \in \{b, a^{\frac{n}{2}} b\}$. In this case, besides the subgroups of the abelian group A, for each divisor r of n, the generalized dicyclic group $Dic_{4n}(A)$ contains the subgroups $H_i^r = \langle a^{\frac{n}{r}}, a^{i-1}\gamma\rangle$, where $i \in \{1, 2, \ldots, \frac{n}{r}\}$. Hence,

$$|L(Dic_{4n}(A))| = |L(A)| + \sigma(n).$$

Since $g^\gamma = g^{-1}$, for all $g \in A$, and A is an abelian group, it is easy to see that the subgroups of A commute with all subgroups of $Dic_{4n}(A)$. Consequently, there are $|L(A)|(|L(A)|+\sigma(n))$ pairs of subgroups of the generalized dicyclic group which commute. Hence,

$$sd(Dic_{4n}(A)) = \frac{|L(A)|(|L(A)| + \sigma(n)) + \sum\limits_{s|n} \sum\limits_{i=1}^{\frac{n}{s}} |C(H_i^s)|}{(|L(A)| + \sigma(n))^2}. \tag{3}$$

Moreover, for a divisor r of n and a fixed $i \in \{1, 2, \ldots, \frac{n}{r}\}$, we have

$$C(H_i^r) = L(A) \cup \left\{ H_j^t \mid H_i^r H_j^t = H_j^t H_i^r,\ t|n \ and \ j \in \left\{1, 2, \ldots, \frac{n}{t}\right\}\right\}.$$

Denote by x, y and e_1 the elements $a\langle\gamma^2\rangle, \gamma\langle\gamma^2\rangle$ and $e\langle\gamma^2\rangle$, respectively. Then

$$\frac{Dic_{4n}(A)}{\langle\gamma^2\rangle} = \langle x, y \mid x^n = y^2 = e_1, x^y = x^{-1}\rangle \cong D_{2n}.$$

Using the same reasoning as in the proof of Theorem 2.1, the subgroups H_i^r and H_j^t of $Dic_{4n}(A)$ commute if and only if the corresponding subgroups, through the above isomorphism, of D_{2n} commute. We obtain

$$\sum_{s|n} \sum_{i=1}^{\frac{n}{s}} |C(H_i^s)| = |L(A)|\sigma(n) + [(m-1)2^{m+3} + 9]g(m').$$

After making the necessary replacements in (3), we finish the proof in this case.

If $m \geq 1$, $m' \neq 1$ and $\gamma^2 = a^{\frac{n}{2}}$, besides the subgroups of A, for each divisor r of n, the generalized dicyclic group $Dic_{4n}(A)$ contains the subgroups $\langle (ab)^{\frac{n}{r}}, (ab)^{i-1}\gamma\rangle$, where $i \in \{1, 2, \ldots, \frac{n}{r}\}$. The steps of the proof are similar with the ones we already made, and we get the same explicit formula for the subgroup commutativity degree of $Dic_{4n}(A)$. Hence, the statement *i)* is proved.

ii) Let $n = 2^m m' \geq 2$, such that $m \geq 2, m' = 1$ and $\gamma^2 = a^{\frac{n}{2}}$. In this case, the generalized dicyclic group is

$$Dic_{4n}(A) = \langle a, b, \gamma \mid a^n = b^2 = \gamma^4 = e, ab = ba, a^\gamma = a^{-1}, b^\gamma = b^{-1}\rangle.$$

Again, the subgroups of the abelian group A commute with all subgroups of $Dic_{4n}(A)$ as a consequence of the equality $g^\gamma = g^{-1}$, for all $g \in A$. Also, for the same reason, the subgroups of $Dic_{4n}(A)$ which are not contained in A commute with the subgroups of A. Since

$$\langle b\rangle \cap \langle a, \gamma \mid a^n = \gamma^4 = e, a^\gamma = a^{-1}\rangle = \{e\}$$

and the two subgroups commute because $ab = ba$ and $b^\gamma = b^{-1}$, we infer that

$$Dic_{4n}(A) \cong \mathbb{Z}_2 \times Q_{2^{m+1}}.$$

Hence, finding an explicit formula for computing the subgroup commutativity degree of the generalized dicyclic group $Dic_{4n}(A)$ is equivalent to the same problem related to the direct product $\mathbb{Z}_2 \times Q_{2^{m+1}}$. Until now, we proved that there are $|L(A)|(2|L(\mathbb{Z}_2 \times Q_{2^{m+1}})| - |L(A)|)$ pairs of subgroups which commute. Further, we need to count the pairs of subgroups of $(L(\mathbb{Z}_2 \times Q_{2^{m+1}}) \setminus L(A))^2$ with the same property.

Denote by x, y, b_1 and e_1 the elements $a\langle\gamma^2\rangle, \gamma\langle\gamma^2\rangle, b\langle\gamma^2\rangle$ and $e\langle\gamma^2\rangle$, respectively. We get the following isomorphism

$$\frac{\mathbb{Z}_2 \times Q_{2^{m+1}}}{\langle\gamma^2\rangle} = \langle b_1\rangle \times \langle x, y \mid x^{2^m} = y^2 = e_1, x^y = x^{-1}\rangle \cong \mathbb{Z}_2 \times D_{2n'},$$

where $n' = \frac{n}{2} = 2^{m-1}$. All subgroups of $L(\mathbb{Z}_2 \times Q_{2^{m+1}}) \setminus L(A)$ contain $\langle\gamma^2\rangle$, so their commutativity is equivalent with the commutativity of the corresponding subgroups of the direct product $\mathbb{Z}_2 \times D_{2n'}$ through the above isomorphism.

These subgroups are divided into the following 5 sets:

$$H_1 = \left\{\langle x^{\frac{n'}{r}}, x^{i-1}y\rangle \mid r|n', i \in \left\{1, 2, \ldots, \frac{n'}{r}\right\}\right\},$$

$$H_2 = \left\{\langle x^{\frac{n'}{r}}b_1, x^{i-1}y\rangle \mid r|n', i \in \left\{1, 2, \ldots, \frac{n'}{r}\right\}\right\},$$

$$H_3 = \left\{\langle x^{\frac{n'}{r}}, x^{i-1}yb_1\rangle \mid r|n', i \in \left\{1, 2, \ldots, \frac{n'}{r}\right\}\right\},$$

$$H_4 = \left\{\langle x^{\frac{n'}{r}}b_1, x^{i-1}yb_1\rangle \mid r \neq 1, r|n', i \in \left\{1, 2, \ldots, \frac{n'}{r}\right\}\right\},$$

$$H_5 = \left\{\langle b_1, x^{\frac{n'}{r}}, x^{i-1}y\rangle \mid r \neq 1, r|n', i \in \left\{1, 2, \ldots, \frac{n'}{r}\right\}\right\}.$$

The general form of one of the above listed subgroups is

$$H_i^r(\alpha, \beta, \delta) = \langle b_1^\alpha, x^{\frac{n'}{r}}b_1^\beta, x^{i-1}yb_1^\delta\rangle,$$

where $r|n', i \in \{1, 2, \ldots \frac{n'}{r}\}$ and $\alpha, \beta, \delta \in \{0, 1\}$. Another useful way to write $H_i^r(\alpha, \beta, \delta)$ is the following one

$$H_i^r(\alpha, \beta, \delta) = H_0^r(\alpha, \beta)\langle x^{i-1}yb_1^\delta\rangle,$$

where $H_0^r(\alpha,\beta) = \langle b_1^\alpha, x^{\frac{n'}{r}} b_1^\beta\rangle$ is a normal subgroup of $\mathbb{Z}_2 \times D_{2n'}$ contained in $\langle b_1, x\rangle \cong \mathbb{Z}_2\times\mathbb{Z}_{n'}$. Taking another subgroup of the same form, namely $H_j^s(\alpha_1,\beta_1,\delta_1) = \langle b_1^{\alpha_1}, x^{\frac{n'}{s}} b_1^{\beta_1}, x^{j-1}yb_1^{\delta_1}\rangle$, where $s|n', j \in \{1,2,\dots,\frac{n'}{s}\}$, $\alpha_1,\beta_1,\delta_1 \in \{0,1\}$, and noticing that $b_1 \in Z(\mathbb{Z}_2 \times D_{2n'})$, we obtain

$$H_i^r(\alpha,\beta,\delta)H_j^s(\alpha_1,\beta_1,\delta_1) = H_j^s(\alpha_1,\beta_1,\delta_1)H_i^r(\alpha,\beta,\delta)$$
$$\Longleftrightarrow x^{2(i-j)} \in H_0^r(\alpha,\beta)H_0^s(\alpha_1,\beta_1). \tag{4}$$

.

The structure of $H_0^r(\alpha,\beta)H_0^s(\alpha_1,\beta_1)$ is changing when we choose two arbitrary subgroups from the sets H_k, where $k \in \{1,2,3,4,5\}$, but we can determine it by inspecting the subgroup lattice of $\mathbb{Z}_2 \times \mathbb{Z}_{n'}$. We denote by $c_{k_1k_2}$ the number of pairs of subgroups which commute and are contained in the sets $H_{k_1} \times H_{k_2}$, where $k_1,k_2 \in \{1,2,3,4,5\}$ with $k_2 \geq k_1$ (since we have a symmetry). Our next purpose is to compute these quantities. We mention that $k_1,k_2 \in \{4,5\}$ implies $r \neq 1$ and $s \neq 1$.

c1) $k_1 = k_2 = 1, k_1 = k_2 = 3$ and $k_1 = 1, k_2 = 3$.

In this case, (4) becomes

$$x^{2(i-j)} \in H_0^r(0,0)H_0^s(0,0) = \langle x^{\frac{n'}{[r,s]}}\rangle \Longleftrightarrow \frac{n'}{[r,s]}|2(i-j), \tag{5}$$

where $i \in \{1,2,\dots,\frac{n'}{r}\}$ and $j \in \{1,2,\dots,\frac{n'}{s}\}$. In [17] (see Subsection 3.1) it was shown that (5) has $2^{m+2}(m-2)+9$ solutions. Hence,

$$c_{k_1k_2} = 2^{m+2}(m-2)+9.$$

c2) $k_1 = k_2 = 2$.

In this situation, (4) is written as

$$x^{2(i-j)} \in H_0^r(0,1)H_0^s(0,1) = \begin{cases} \langle b_1, x^{\frac{n'}{[r,s]}}\rangle & \text{, if } r \neq s \\ \langle x^{\frac{n'}{r}} b_1\rangle & \text{, if } r = s \end{cases}.$$

If $r = s = 1$, then $i \equiv j(mod\ \frac{n'}{2})$, where $i,j \in \{1,2,\dots,n'\}$. For each i, the congruence has 2 solutions $j \in \{1,2,\dots,n'\}$. If $r = s > 1$, then $i \equiv j(mod\ \frac{n'}{r})$, where $i,j \in \{1,2,\dots,\frac{n'}{r}\}$. For each i, the congruence has only one solution. We get a total of

$$2n' + \sum_{\substack{r|n'\\ r>1}} \frac{n'}{r} = n' + \sigma(n')$$

pairs of subgroups which commute, corresponding to the case $r = s$. If $r \neq s$, we must count the number of solutions of (5). If $r = s = n'$, we find only one solution. If $r = s < n'$, then (5) is equivalent to $i \equiv j(mod\ \frac{n'}{2r})$, where $i, j \in \{1, 2, \ldots, \frac{n'}{r}\}$. For each i, this last congruence has 2 solutions. Hence, for $r = s$, we have

$$1 + \sum_{\substack{r|n' \\ r \neq n'}} 2\frac{n'}{r} = 2\sigma(n') - 1$$

solutions. Consequently, to find the pairs of subgroups which commute in the case $r \neq s$, we must subtract the above number from $2^{m+2}(m-2) + 9$. We obtain

$$c_{k_1k_2} = n' + \sigma(n') + 2^{m+2}(m-2) + 9 - 2\sigma(n') + 1 = 2^{m+2}(m-2) - 2^{m-1} + 11.$$

c3) $k_1 = 1, k_2 = 2$ and $k_1 = 2, k_2 = 3$.

Now, (4) becomes

$$x^{2(i-j)} \in H_0^r(0,0)H_0^s(0,1) = \begin{cases} \langle x^{\frac{n'}{s}} b_1 \rangle & , \text{if } r < s \\ \langle b_1, x^{\frac{n'}{[r,s]}} \rangle & , \text{if } r \geq s \end{cases}.$$

Let $r = 2^u$ and s be divisors of n' such that $r < s$. We infer that $u \in \{0, 1, \ldots, m-2\}$ and $i \equiv j(mod\ \frac{n'}{s})$, where $i \in \{1, 2, \ldots, \frac{n'}{r}\}$ and $j \in \{1, 2, \ldots, \frac{n'}{s}\}$. For each i the congruence has one solution. There are $\tau(n') - (u+1)$ divisors s of n' which are greater than the divisor r of n'. Consequently, we get

$$\sum_{u=0}^{m-2} 2^{m-1-u}(\tau(n') - (u+1)) = \sum_{t=1}^{m-1} 2^t t = 2^m(m-2) + 2$$

pairs of subgroups which commute corresponding to the case $r < s$. We already saw that, for $r = s$, (5) has $2\sigma(n') - 1$ solutions. Now, we need to discuss the case $r > s$. If $r = n'$, then we obtain

$$\sum_{\substack{s|n' \\ s \neq n'}} \sum_{j=1}^{\frac{n'}{s}} 1 = \sigma(n') - 1$$

more solutions for (5). Let $r = 2^u$ and $s = 2^v$ be divisors of n' such that $r > s$ and $r \neq n'$. It follows that $u \in \{1, 2, \ldots, m-2\}$ and $v \in \{0, 1, \ldots, m-3\}$. We can rewrite (5) as $i \equiv j(mod\ 2^{m-2-u})$. If r is fixed, this congruence has

$$\sum_{v=0}^{u-1} 2^{u-v+1} = 2^{u+2}\left[1 - \left(\frac{1}{2}\right)^u\right]$$

solutions for each i. Consequently, if $r \geq s$, we obtain

$$3\sigma(n') - 2 + \sum_{u=1}^{m-2} 2^{m+1}\left[1 - \left(\frac{1}{2}\right)^u\right] = 3\sigma(n') - 2 + 2^{m+1}[(m-2) + 2^{2-m} - 1]$$

pairs of subgroups which commute. We conclude that

$$c_{k_1k_2} = 3\sigma(n') + 2^m(m-2) + 2^{m+1}[(m-2) + 2^{2-m} - 1] = 2^m(3m-5) + 5.$$

c4) $k_1 = 1, k_2 = 4$ and $k_1 = 3, k_2 = 4$.

Then, (4) states that

$$x^{2(i-j)} \in H_0^r(0,0)H_0^s(0,1) = \begin{cases} \langle x^{\frac{n'}{s}} b_1\rangle & \text{, if } r < s \\ \langle b_1, x^{\frac{n'}{[r,s]}}\rangle & \text{, if } r \geq s \end{cases}.$$

The steps that must be performed to analyse this case are similar with the ones we made to solve *c3)*, the only change being that $s \neq 1$. We obtain

$$c_{k_1k_2} = 3(\sigma(n') - n') + 2^m(m-2) + 2^m[(m-2) + 2(2^{2-m} - 1)] = 2^{m-1}(4m-9) + 5.$$

c5) $k_1 = 1, k_2 = 5$ and $k_1 = 3, k_2 = 5$.

In this situation, (4) can be written as

$$x^{2(i-j)} \in H_0^r(0,0)H_0^s(1,0) = \langle b_1, x^{\frac{n'}{[r,s]}}\rangle,$$

which is equivalent to counting the solutions of (5), where r and s are divisors of n', $s \neq 1$ and $i \in \{1, 2, \dots, \frac{n'}{r}\}, j \in \{1, 2, \dots, \frac{n'}{s}\}$. If $r = n'$, then (5) has

$$\sum_{\substack{s|n' \\ s\neq 1}} \sum_{j=1}^{\frac{n'}{s}} 1 = \sigma(n') - n'$$

solutions. If $s = n'$ and $r \neq n'$ is fixed, (5) has only one solution for each i. Take $r = 2^u$ and $s = 2^v$, where $u \in \{0, 1, \dots, m-2\}$ and $v \in \{1, 2, \dots, m-2\}$. Then, (5) is equivalent to the congruence $i \equiv j (mod\ 2^{max\{u,v\}-v+1})$. For a fixed r, this congruence has

$$1 + \sum_{v=1}^{m-2} 2^{max\{u,v\}-v+1} = 2^{u+1} - 2u + 2m - 5$$

solutions for each i. Hence, the total number of pairs of subgroups which commute is

$$\begin{aligned} c_{k_1k_2} &= \sigma(n') - n' + \sum_{u=0}^{m-2} 2^{m-1-u}(2^{u+1} - 2u + 2m - 5) \\ &= \sigma(n') - n' + 2^m(3m-8) + 10 \\ &= 2^{m-1}(6m-15) + 9. \end{aligned}$$

c6) $k_1 = 2, k_2 = 4$.

In this case, (4) becomes

$$x^{2(i-j)} \in H_0^r(0,1)H_0^s(0,1) = \begin{cases} \langle b_1, x^{\frac{n'}{s}} \rangle & \text{, if } r = 1 \\ \langle b_1, x^{\frac{n'}{[r,s]}} \rangle & \text{, if } r \neq s \text{ and } r \neq 1 \\ \langle x^{\frac{n'}{s}} b_1 \rangle & \text{, if } r = s \end{cases}.$$

We use some results that are already proved and edit them since $s \neq 1$. According to *c2)*, if $r = s$, we have $\sigma(n') - n'$ pairs of subgroups which commute. For $r \neq s$ and $r \neq 1$, by inspecting *c3)* and *c4)*, we get

$$2\{(\sigma(n') - n' - 1) + 2^m[(m-2) + 2(2^{2-m} - 1)]\}$$

more such pairs. Let $r = 1$. Then, once again we get (5), where $i \in \{1, 2, \ldots, n'\}$ and $j \in \{1, 2, \ldots, \frac{n'}{s}\}$, which provides $n' + 2n'(m-2)$ pairs of subgroups which commute. Adding up the obtained numbers, it follows that

$$c_{k_1k_2} = 3\sigma(n') + 2n'(m-3) - 2 + 2^{m+1}[(m-2) + 2(2^{2-m} - 1)] = 2^m(3m-8) + 11.$$

c7) $k_1 = 2, k_2 = 5$.

Now, (4) states that

$$x^{2(i-j)} \in H_0^r(0,1)H_0^s(1,0) = \begin{cases} \langle b_1, x^{\frac{n'}{s}} \rangle & \text{, if } r = 1 \\ \langle b_1, x^{\frac{n'}{[r,s]}} \rangle & \text{, if } r \neq 1 \end{cases}.$$

Again, we use our work done in the previous cases. Let $r \neq 1$. By *c2)*, if $r = s$, (5) has $1 + 2(\sigma(n') - n' - 1)$ solutions. According to *c3)* and *c4)*, for $r \neq s$, (5) has

$$2\{(\sigma(n') - n' - 1) + 2^m[(m-2) + 2(2^{2-m} - 1)]\}$$

more solutions. Also, *c6)* provides the number of pairs of subgroups which commute if $r = 1$, this being $n' + 2n'(m-2)$. Hence,

$$\begin{aligned} c_{k_1k_2} &= 1 + 4(\sigma(n') - n' - 1) + n' + 2n'(m-2) + 2^{m+1}[(m-2) + 2(2^{2-m} - 1)] \\ &= 2^{m-1}(6m - 15) + 9. \end{aligned}$$

c8) $k_1 = k_2 = 4$.

Then, (4) becomes

$$x^{2(i-j)} \in H_0^r(0,1)H_0^s(0,1) = \begin{cases} \langle x^{\frac{n'}{r}} b_1 \rangle & \text{, if } r = s \\ \langle b_1, x^{\frac{n'}{[r,s]}} \rangle & \text{, if } r \neq s \end{cases}.$$

According to *c2)*, *c3)* and *c4)*, we obtain

$$\begin{aligned} c_{k_1k_2} &= \sigma(n') - n' + 2(\sigma(n') - n' - 1) + 2^{m+1}[(m-2) + 2(2^{2-m} - 1)] \\ &= 2^{m-1}(4m - 13) + 11. \end{aligned}$$

c9) $k_1 = 4, k_2 = 5$ and $k_1 = k_2 = 5$.

In this case, (4) becomes

$$x^{2(i-j)} \in H_0^r(0,1)H_0^s(1,0) = \langle b_1, x^{\frac{n'}{[r,s]}} \rangle.$$

Following the same arguments as the ones that were used to solve *c7)*, the number of pairs of subgroups which commute is

$$c_{k_1k_2} = 1+4(\sigma(n')-n'-1)+2^{m+1}[(m-2)+2(2^{2-m}-1)] = 2^m(2m-6)+9.$$

Consequently, the number of pairs of subgroups which commute, contained in the sets $H_{k_1} \times H_{k_2}$, where $k_1, k_2 \in \{1,2,3,4,5\}$, is

$$4c_{11} + c_{22} + 4c_{12} + 4c_{14} + 4c_{15} + 2c_{24} + 2c_{25} + c_{44} + 3c_{45} = 2^m(72m - 164) + 201.$$

Hence, the subgroup commutativity degree of $\mathbb{Z}_2 \times Q_{2^{m+1}}$ is given by

$$sd(\mathbb{Z}_2 \times Q_{2^{m+1}}) = \frac{|L(A)|(2|L(\mathbb{Z}_2 \times Q_{2^{m+1}})| - |L(A)|) + 2^m(72m-164)+201}{|L(\mathbb{Z}_2 \times Q_{2^{m+1}})|^2}.$$

Recall that the subgroups included in the sets H_k, with $k \in \{1,2,3,4,5\}$, are the images of the subgroups of $\mathbb{Z}_2 \times Q_{4n'}$, which are not contained in the abelian group A, through the isomorphism $\frac{\mathbb{Z}_2 \times Q_{4n'}}{\langle \gamma^2 \rangle} \cong D'_{2n}$. Their number is $5\sigma(n') - 2n'$. Corollary 2.3 leads us to $|L(A)| = 3\tau(n') + 2 = 3m + 2$. Since the total number of subgroups of $\mathbb{Z}_2 \times Q_{2^{m+1}}$ was previously indicated (see (2)), we obtain the explicit formula

$$sd(\mathbb{Z}_2 \times Q_{2^{m+1}}) = \frac{2^{m+2}(24m-37) + 9m^2 - 18m + 185}{[2^{m+2} + 3(m-1)]^2}.$$

iii) Let $n = 2$ and $\gamma^2 \in \{a^{\frac{n}{2}}, b, a^{\frac{n}{2}}b\}$. Since, in this case, $Dic_{4n}(A)$ is an abelian group of order 8 generated by an element of order 2 and an element of order 4, the conclusion follows. □

As a consequence of Theorem 2.4, we finish this section by providing another example of a class of groups whose subgroup commutativity degree vanishes asymptotically.

Corollary 2.5. $\lim_{m\to\infty} sd(\mathbb{Z}_2 \times Q_{2^{m+1}}) = 0.$

3. Cyclic subgroup commutativity degree of (generalized) dicyclic groups. We apply similar methods as the ones we used in Section 2 to obtain an explicit formula for the cyclic subgroup commutativity degree of (generalized) dicyclic groups.

To determine the cyclic subgroup commutativity degree of the dicyclic group Dic_{4n}, the first step is to write down its cyclic subgroups. For each divisor r of $2n$, the dicyclic group has one cyclic subgroup isomorphic to $\mathbb{Z}_r$, namely $H_0^r = \langle a^{\frac{2n}{r}} \rangle$. Also, the subgroups $H_i^1 = \langle a^n, a^{i-1}\gamma \rangle$, where $i \in \{1,2,\ldots,n\}$, are cyclic since $(a^{i-1}\gamma)^2 = a^n$, for all $i \in \{1,2,\ldots,n\}$.

THEOREM 3.1. *Let $n \geq 2$ be a positive integer. The cyclic subgroup commutativity degree of the dicyclic group Dic_{4n} is*

$$csd(Dic_{4n}) = \begin{cases} \frac{\tau(2n)(\tau(2n)+n)+n(\tau(2n)+1)}{(\tau(2n)+n)^2} & , \text{ if } n \equiv 1 \ (mod\ 2) \\ \\ \frac{\tau(2n)(\tau(2n)+n)+n(\tau(2n)+2)}{(\tau(2n)+n)^2} & , \text{ if } n \equiv 0 \ (mod\ 2) \end{cases}.$$

In particular, we have

$$csd(Dic_{12}) = \frac{43}{49} \text{ and } csd(Dic_{16}) = \frac{7}{8}.$$

Proof. The cyclic subgroup commutativity degree of Dic_{4n} is given by

$$csd(Dic_{4n}) = \frac{1}{|L_1(Dic_{4n})|^2}\left(\sum_{r|2n} |C_1(H_0^r)| + \sum_{i=1}^{n} |C_1(H_i^1)|\right). \tag{6}$$

Since we specified which are the cyclic subgroups of Dic_{4n}, we can count them and obtain

$$|L_1(Dic_{4n})| = \tau(2n) + n.$$

In the proof of Theorem 2.1, we remarked that for each divisor r of $2n$, the subgroup H_0^r commutes with all subgroups of Dic_{4n}. It follows that

$$\sum_{r|2n} |C_1(H_0^r)| = \tau(2n)|L_1(Dic_{4n})| = \tau(2n)(\tau(2n) + n).$$

Let $i \in \{1, 2, \ldots, n\}$ and consider the cyclic subgroup H_i^1 of Dic_{4n}. We have

$$|C_1(H_i^1)| = \left(\bigcup_{r|2n} \{H_0^r\}\right) \cup \left\{H_j^1 \mid H_i^1 H_j^1 = H_j^1 H_i^1, where\ j \in \{1, 2, \ldots, n\}\right\}.$$

Using the isomorphism $\frac{Dic_{4n}}{Z(Dic_{4n})} \cong D_{2n}$, we infer that the subgroups H_i^1 and H_j^1 of Dic_{4n} commute if and only if the subgroups $K_i^1 = \langle x^{i-1}y\rangle$ and $K_j^1 = \langle x^{j-1}y\rangle$ of D_{2n} commute. In [20] (see Subsection 3.2), it was proved that $K_j^1 \in C(K_i^1)$ if and only if $j = i$ and n is odd, or, $j \in \{i, (i + \frac{n}{2})(mod\ n)\}$ and n is even. Hence,

$$\sum_{i=1}^{n} |C_1(H_i^1)| = \begin{cases} n(\tau(2n) + 1) & , \text{ if } n \equiv\ 1\ (mod\ 2) \\ n(\tau(2n) + 2) & , \text{ if } n \equiv\ 0\ (mod\ 2) \end{cases}.$$

After making the replacements in (6), we obtain the desired formula and finish our proof. ◻

We end our study by finding an explicit formula for the cyclic subgroup commutativity degree of the generalized dicyclic groups $Dic_{4n}(A)$, where $A \cong \mathbb{Z}_2 \times \mathbb{Z}_n$.

Theorem 3.2. *Let $A \cong \mathbb{Z}_2 \times \mathbb{Z}_n$ be an abelian group, where $n = 2^m m' \geq 2$, $m \in \mathbb{N}^*$ and m' is a positive odd integer.*

i) If $m \geq 1, m' \neq 1$ and $\gamma^2 \in \{a^{\frac{n}{2}}, b, a^{\frac{n}{2}}b\}$, or, if $m \geq 2, m' = 1$ and $\gamma^2 \in \{b, a^{\frac{n}{2}}b\}$, the cyclic subgroup commutativity degree of the generalized dicyclic group $Dic_{4n}(A)$ is

$$csd(Dic_{4n}(A)) = \frac{|L_1(A)|(|L_1(A)| + n) + n(|L_1(A)| + 2)}{(|L_1(A)| + n)^2}.$$

In particular, we have

$$csd(Dic_{4n}(\mathbb{Z}_2 \times \mathbb{Z}_6)) = \frac{43}{49}.$$

ii) If $m \geq 2, m' = 1$ and $\gamma^2 = a^{\frac{n}{2}}$, the generalized dicyclic group $Dic_{4n}(A)$ is isomorphic to the direct product $\mathbb{Z}_2 \times Q_{2^{m+1}}$ and its cyclic subgroup commutativity degree is

$$csd(\mathbb{Z}_2 \times Q_{2^{m+1}}) = \frac{2^m(4m + 8) + (2m + 2)^2}{(2^m + 2m + 2)^2}.$$

In particular, we have

$$csd(\mathbb{Z}_2 \times Q_{16}) = \frac{224}{256}.$$

iii) If $m = m' = 1$ and $\gamma^2 \in \{a^{\frac{n}{2}}, b, a^{\frac{n}{2}}b\}$, the generalized dicyclic group $Dic_{4n}(A)$ is isomorphic to the abelian group $\mathbb{Z}_2 \times \mathbb{Z}_4$ and its cyclic subgroup commutativity degree is

$$csd(\mathbb{Z}_2 \times \mathbb{Z}_4) = 1.$$

Proof. *i)* Let $n = 2^m m' \geq 2$, such that $m \geq 1, m' \neq 1$, or, $m \geq 2, m' = 1$, and $\gamma^2 \in \{b, a^{\frac{n}{2}}b\}$. The cyclic subgroups of $Dic_{4n}(A)$ are those contained in the abelian group A and the subgroups $H_i^1 = \langle a^{i-1}\gamma\rangle$, where $i \in \{1, 2, \ldots, n\}$. This leads us to

$$|L_1(Dic_{4n}(A))| = |L_1(A)| + n.$$

In the proof of statement *i)* of Theorem 2.4, we showed that $\frac{Dic_{4n}(A)}{\langle\gamma^2\rangle} \cong D_{2n}$. Once again, through this isomorphism, the subgroup H_i^1 has a correspondent subgroup $K_i^1 = \langle x^{i-1}y\rangle$ of the dihedral group D_{2n}, for each $i \in \{1, 2, \ldots, n\}$. As we saw in the proof of Theorem 3.1, since n is even, K_i^1 commutes with 2 cyclic subgroups of D_{2n}. Consequently, H_i^1 commutes with two cyclic subgroups of $Dic_{4n}(A)$ that contain $\langle\gamma^2\rangle$. Also, since all cyclic subgroups contained in A commute with all cyclic subgroups of $Dic_{4n}(A)$, we conclude that

$$csd(Dic_{4n}(A)) = \frac{|L_1(A)|(|L_1(A)| + n) + n(|L_1(A)| + 2)}{(|L_1(A)| + n)^2}.$$

The same reasoning is used if $m \geq 1, m' \neq 1$ and $\gamma^2 = a^{\frac{n}{2}}$, the only change being that, besides the cyclic subgroups contained in A, the generalized dicyclic group has other n cyclic subgroups, namely $\langle(ab)^{i-1}\gamma\rangle$, where $i \in \{1, 2, \ldots, n\}$.

ii) Let $n = 2^m m' \geq 2$, such that $m \geq 2, m' = 1$ and $\gamma^2 = a^{\frac{n}{2}}$. Denote by n' the quantity 2^{m-1}. A way to show the isomorphism $Dic_{4n}(A) \cong \mathbb{Z}_2 \times Q_{2^{m+1}}$ was illustrated in the proof of Theorem 2.4, *ii)*. To study the commutativity of the cyclic subgroups of $\mathbb{Z}_2 \times Q_{2^{m+1}}$, we must check the sets H_k, where $k \in \{1, 2, 3, 4, 5\}$, which were listed in the same proof. We remark that besides the cyclic subgroups contained in A, the direct product $\mathbb{Z}_2 \times Q_{2^{m+1}}$ has $2n'$ cyclic subgroups that correspond to some cyclic subgroups of $\mathbb{Z}_2 \times D_{2n'}$, namely $\{\langle x^{i-1}y\rangle \mid i \in \{1, 2, \ldots, n'\}\} \subset H_1$ and $\{\langle x^{i-1}yb_1\rangle \mid i \in \{1, 2, \ldots, n'\}\} \subset H_3$. Consequently,

$$|L_1(\mathbb{Z}_2 \times Q_{2^{m+1}})| = |L_1(A)| + 2n'.$$

To see how many pairs of cyclic subgroups contained in the sets $H_{k_1} \times H_{k_2}$ commute, where $k_1, k_2 \in \{1, 3\}$, we move our attention to the case *c1)* that was examined in the proof of Theorem 2.4, *ii)*. We infer that we must count the solutions of the congruence $i \equiv j (mod\ \frac{n'}{2})$, where $i, j \in \{1, 2, \ldots, n'\}$. It is easy to see that for each i, this congruence has 2 solutions. Hence, there are $8n'$ pairs of cyclic subgroups which commute and are contained in the sets $H_{k_1} \times H_{k_2}$, where $k_1, k_2 \in \{1, 3\}$. Besides them, since the cyclic subgroups contained in A commute with all cyclic subgroups of $\mathbb{Z}_2 \times Q_{2^{m+1}}$, we get other $|L_1(A)|(|L_1(A)| + 4n')$ such pairs. Then the cyclic subgroup commutativity degree of $\mathbb{Z}_2 \times Q_{2^{m+1}}$ is

$$csd(\mathbb{Z}_2 \times Q_{2^{m+1}}) = \frac{|L_1(A)|(|L_1(A)| + 2^{m+1}) + 2^{m+2}}{(|L_1(A)| + 2^m)^2}. \tag{7}$$

According to Theorem 12 of [19], the number of cyclic subgroups of $A \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^m}$ is

$$|L_1(A)| = 2m + 2.$$

Once we make the necessary replacements in (7), we obtain

$$csd(\mathbb{Z}_2 \times Q_{2^{m+1}}) = \frac{2^m(4m + 8) + (2m + 2)^2}{(2^m + 2m + 2)^2}.$$

iii) Let $n = 2$. The isomorphism $Dic_8(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ holds (see the proof of Theorem 2.4, *iii)*). Consequently, the conclusion follows. □

Our final result indicates that the cyclic subgroup commutativity degree of $\mathbb{Z}_2 \times Q_{2^{m+1}}$ tends to 0 as the order of the group tends to infinity.

Corollary 3.3. $\lim_{m \to \infty} csd(\mathbb{Z}_2 \times Q_{2^{m+1}}) = 0.$

4. Further research. Since there is a growing interest in probabilistic aspects of finite group theory, we indicate four open problems related to the (cyclic) subgroup commutativity degree, which may constitute the subject of some further research.

Problem 4.1. For a fixed $\alpha \in (0,1)$, describe the structure of finite groups G satisfying $sd(G) = (\leq,\geq)\alpha$ or $csd(G) = (\leq,\geq)\alpha$. What can be said about two finite groups having the same (cyclic) subgroup commutativity degree?

Problem 4.2. Which are the connections between the (cyclic) subgroup commutativity degree of a finite group and the (cyclic) subgroup commutativity degrees of its subgroups or quotients?

Problem 4.3. Let G be a finite group. Study the properties of the maps $sd : L(G) \longrightarrow [0,1]$, $H \mapsto sd(H)$, and $csd : L(G) \longrightarrow [0,1], H \mapsto csd(H)$.

Problem 4.4. Compute explicitly the (cyclic) subgroup commutativity degree of the generalized dicyclic group $Dic_{4n}(A)$, where A is an arbitrary abelian group of order $2n$.

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